Birth-death processes

- Steady-state probabilities:

$$d_0 = 1 \qquad d_j = \frac{\lambda_0 \lambda_1 \cdots \lambda_{j-1}}{\mu_1 \mu_2 \cdots \mu_j} \quad \text{for } j = 1, 2, \ldots \qquad D = \sum_{i=0}^{\infty} d_i \qquad \pi_j = \frac{d_j}{D} \quad \text{for } j = 0, 1, 2, \ldots$$

- Expected number of customers in the system: $\ell = \sum_{n=0}^{\infty} n \pi_n$

- Expected number of customers in the queue, s parallel servers: $\ell_q = \sum_{n=s+1}^{\infty} (n - s) \pi_n$

- Effective arrival rate: $\lambda_{\text{eff}} = \sum_{i=0}^{\infty} \lambda_i \pi_i$

- Little's law (system-wide): $\ell = \lambda_{\text{eff}} w$

- Little's law (queue only): $\ell_q = \lambda_{\text{eff}} w_q$

Standard queueing models

Y_t ~ Poisson random variable with parameter λt:

$$p_{Y_t}(n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!} \quad \text{for } n = 0, 1, 2, \ldots \qquad E[Y_t] = \lambda t$$

M/M/∞:

- Steady-state probabilities: $\pi_j = \Pr\{L = j\}$ where L is a Poisson random variable with parameter λ/μ

M/M/s:

- Steady-state probabilities:

$$\rho = \frac{\lambda}{s\mu} \qquad \pi_0 = \left[\left(\sum_{j=0}^{s} \frac{(s\rho)^j}{j!}\right) + \frac{s^s \rho^{s+1}}{s!(1-\rho)}\right]^{-1} \qquad \pi_j = \begin{cases} \dfrac{(\lambda/\mu)^j}{j!}\pi_0 & \text{for } j = 1, 2, \ldots, s \\ \dfrac{(\lambda/\mu)^j}{s! s^{j-s}}\pi_0 & \text{for } j = s+1, s+2, \ldots \end{cases}$$

- Expected number of customers in queue: $\ell_q = \dfrac{\pi_s \rho}{(1-\rho)^2}$

- Expected number of customers in the system: $\ell = \ell_q + \dfrac{\lambda}{\mu}$

G/G/s:

- Whitt's approximation:

$$G = \text{generic interarrival time random variable with rate } \lambda = \frac{1}{E[G]}$$

$$X = \text{generic service time random variable with rate } \mu = \frac{1}{E[X]}$$

$$\varepsilon_a = \frac{\text{Var}[G]}{E[G]^2} \qquad \varepsilon_s = \frac{\text{Var}[X]}{E[X]^2} \qquad \hat{w}_q \approx \frac{\varepsilon_a + \varepsilon_s}{2} w_q$$